Phase locking in on-off intermittency

Meng Zhan,[1] Gang Hu,[2,1] Dai-Hai He,[1] and Wen-Qi Ma[3]
[1]*Department of Physics, Beijing Normal University, Beijing 100875, China*
[2]*China Center for Advanced Science and Technology (CCAST) (World Laboratory), P. O. Box 8730, Beijing 100080, China*
[3]*Department of Physics, Beihua University, Jilin 132011, China*
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Dynamical behavior of on-off intermittency around chaos synchronization-desynchronization bifurcation parameter line is investigated in coupled identical chaotic oscillators. Along this parameter line, we find that on-off intermittency can transit from phase-unlocking status to phase-locking one in the phase space of variable differences, which can be regarded as a codimension-two bifurcation, i.e., combinative bifurcations of desynchronization and phase locking. In the phase-locking case, the motions of all oscillators are chaotic and they show on-off intermittency with respect to the synchronous manifold, however, spatial phase order of variable differences is clearly established.

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The problem of chaos synchronization has attracted great interest among physicists over the past decade since the pioneering work by Pecora and Carroll [1]. In particular, there has been much interest in using chaos synchronization to understand complex dynamics that arises in diverse systems with interacting nonlinear dynamical units [2]. Very recently, a systematical analysis on the instability of synchronous chaotic state has been carried out [3,4].

Related to chaos synchronization, some important concepts, such as bubbling bifurcation [5], riddled basins [6], and on-off intermittency [7], have been proposed. As a synchronous chaotic state loses its stability, and if there is no other attractor away from the synchronous manifold in the phase space, we can usually observe the phenomenon of on-off intermittency with the "off" state near the synchronous chaotic state and the "on" state showing random bursts of large desynchronization. The bifurcation from stable synchronous chaos to on-off intermittency has been extensively investigated [7]. However, the features of the motion of on-off intermittency have not been studied further. In particular, the behavior of the desynchronous parts of various chaotic oscillators after the instability of synchronous chaos is not well understood. In this paper we will study this problem on the variable difference space. Then we find that there is some interesting spatial order existing in the seemly random on-bursts, and we find a new kind of bifurcation occurring on the on-off bifurcation condition, i.e., a bifurcation from phase-unlocking on-off intermittency to phase-locking one, which can be called a codimension-two bifurcation to on-off-phase-locking intermittency.

Our model is a system of N identical nonlinear oscillators with periodic boundary conditions and couplings between nearest neighbors,

$$\dot{x}_j=\sigma(y_j-x_j),$$

$$\dot{y}_j=\rho x_j-y_j-x_j z_j+(e+r)(x_{j+1}-x_j)+(e-r)(x_{j-1}-x_j), \quad (1)$$

$$\dot{z}_j=x_j y_j-\beta z_j, \quad j=1,\ldots,N,$$

where the single oscillator is the Lorenz model capable of exhibiting chaotic solutions (we use parameters: $\sigma=10$, $\beta=1.0$, and $\rho=28.0$, at which the single Lorenz oscillator is chaotic), and e and r are scalar symmetric and asymmetric coupling parameters [4], respectively. Here we set the system size $N=6$. In fact, we can get the similar result for other N numbers and for different types of couplings.

In Fig. 1(a), we specify the region of chaos synchronization for above coupled model [Eqs. (1)] in the (e,r) coupling parameter plane with "S" region representing stable synchronous chaos while "U" region corresponding to chaos desynchronization. The stable (unstable) region is determined by the negative (positive) largest transversal Lyapunov exponent of the synchronous chaotic state [4]. Bifurcation from synchronous chaos to on-off intermittency occurs on the boundary separating the "S" and "U" regions. Our aim is to investigate the possible characteristic change on this on-off intermittency bifurcation line, or specifically, we go further to study the possible bifurcation point on the bifurcation line of Fig. 1(a).

In order to discuss the behavior of on-off intermittency, we will work in the variable difference space

$$\Delta x_j(t)=x_j(t)-\frac{1}{N}\sum_{j=1}^{N} x_j(t), \quad j=1,\ldots,N. \quad (2)$$

By subtracting the spatial average, all variables keep its desynchronized parts and the synchronous chaos is eliminated. Slightly above the S-U boundary of Fig. 1(a), different oscillators may have rather different motions of $\Delta x_j(t)$ for their desynchronous elements after the instability of the homogeneous chaos. We are interested in the features of the $\Delta x_j(t)$ evolutions in the system state of on-off intermittency.

In Figs. 1(b), 1(c), and 1(d), we display the motions of an arbitrary oscillator Δx_6 at three parameter sets, which correspond to the circle ($e=3.1, r=0.0$), square ($e=9.0, r=3.7$), and triangle ($e=15.1, r=6.0$) in Fig. 1(a), respectively. In all these three cases the figures perspicuously indicate the characteristics of on-off intermittency: irregularity and random

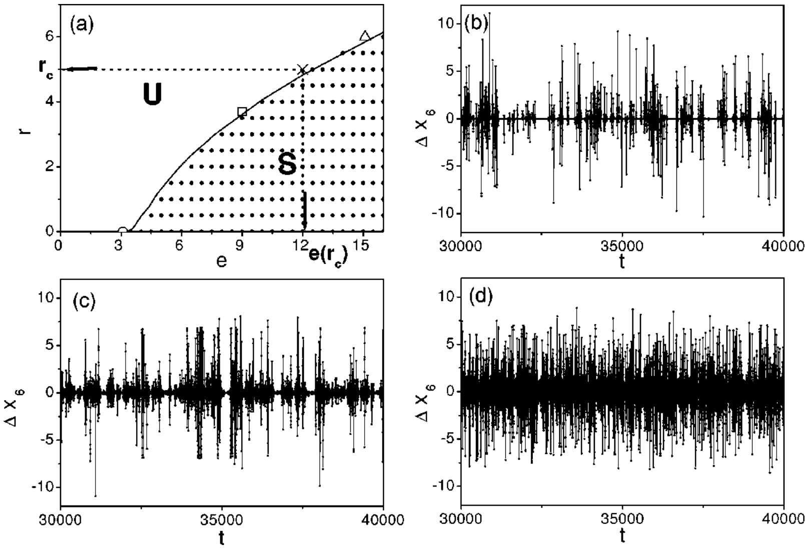

FIG. 1. (a) The region of stable chaos synchronization (S) and that of chaos desynchronization (U) in the coupling parameter (e,r) plane. The system behaviors at the parameter sets $e=3.1, r=0.0$ (circle), $e=9.0, r=3.7$ (square), and $e=15.1, r=6.0$ (triangle) will be investigated in detail. The parameter set $e(r_c)=12.0, r_c=5.0$ (cross) corresponds to the transition point from phase-unlocking state to phase-locking one. (b), (c), and (d) The on-off intermittencies of Δx_6 vs t at the parameter sets, circle, square, and triangle, respectively. Throughout the paper, the integral time step $h=0.01$ is used. But, here for each 100 data we plot one for clearly showing the system behavior in large time scale.

bursts. However, below we will find out some regularity in these seemly random motions, and find a certain bifurcation manifested by the phase of these variable differences.

In order to define the phase of the variable differences, we take the Hilbert transform of Δx_j and define $H(\Delta x_j)$ as

$$H[\Delta x_j(t)] = PV\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Delta x_j(\tau)}{t-\tau}d\tau\right), \tag{3}$$

where PV stands for the Cauchy principal value of the integral. Now we have constructed our two-dimensional variable difference plane $[\Delta x_j, H(\Delta x_j)]$ for each oscillator, and then the phase of the variable difference can be uniquely defined accordingly as

$$\sin\theta_j(t) = \frac{H[\Delta x_j(t)]}{\sqrt{[\Delta x_j(t)]^2 + [H(\Delta x_j(t))]^2}}. \tag{4}$$

This technique of signal processing was used in Ref. [8] to study phase synchronization. Here we apply the same technique to show the rotation of the desynchronous part of motion separated from the synchronous one in Eq. (2), that is, to study the motion of the variable differences. The phase in this chaotic motion can be computed along the normal approach for the Rössler attractor based on the partial Poincare surface-of-section technique [8,9], that is, crossing the secant surface [we choose $H(\Delta x)=0$, and $\Delta x>0$] with clockwise and counterclockwise directions, we get the phase shift -2π and 2π, respectively. Thus, $\theta_j(t)$ can vary to any value [beyond the interval $(0,2\pi)$] and with both Eq. (4) and the above technique there is no any ambiguity for determining phase.

In Fig. 2, we take a set of coupling parameters, $e=3.1, r=0.0$, which is denoted by the circle in Fig. 1(a), and show the dynamical behavior of the system in the variable difference space. In Fig. 2(a) we take time step $h=0.01$ for the numerics and plot the $\Delta x_6(t)-H[\Delta x_6(t)]$ trajectory at discrete times $t=n\Delta T$, $\Delta T=0.1$ (in our system the period for a single Lorenz oscillator to make an oscillation loop is about 1.14). The scattering points represent random bursts of on-off intermittency. In this figure we also plot the positions of $\Delta x_j - H(\Delta x_j)$, $j=1,2\ldots,6$ at an arbitrary time instant. In Fig. 2(b) we plot the phase differences $\Delta\theta_{j,1}$ vs t with

$$\Delta\theta_{j,1}(t) = \theta_j(t) - \theta_1(t), \quad j=2,3,\ldots,6, \tag{5}$$

where $\theta_j(t)$ is given in Eq. (4). It is interesting to see that all $\Delta\theta_{j,1}(t)$'s perform random walks, i.e., they can stay at cer-

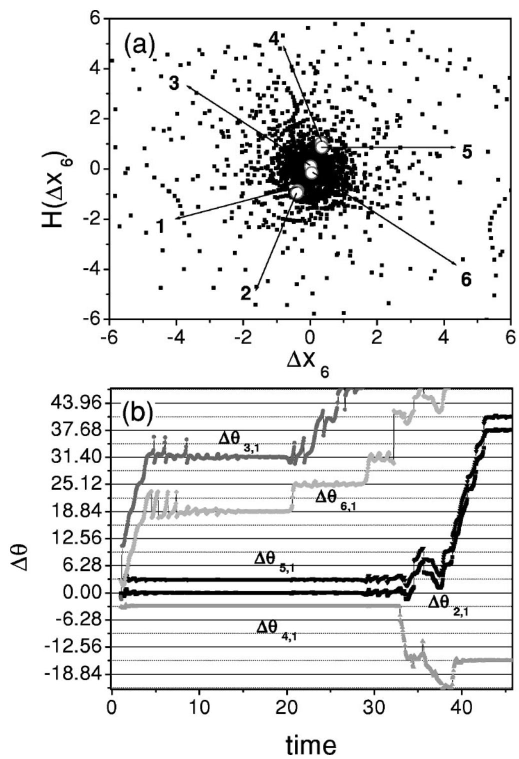

FIG. 2. $e=3.1, r=0.0$ [circle in Fig. 1(a)]. (a) The on-off intermittent trajectory is plotted in $[\Delta x_6,\ H(\Delta x_6)]$ plane at discrete times $t=n\Delta T, \Delta T=0.1, n=1,2,\ldots$, that is, we plot one point each for ten data. The circles in the figure indicate the positions of $\Delta x_j(t), H[\Delta x_j(t)]$ at an arbitrary time. No phase locking is observed. (b) The phase differences of $\Delta\theta_{j,1}$ given by Eq. (4) plotted vs t. $\Delta\theta_{j,1}$ show a behavior of random walk.

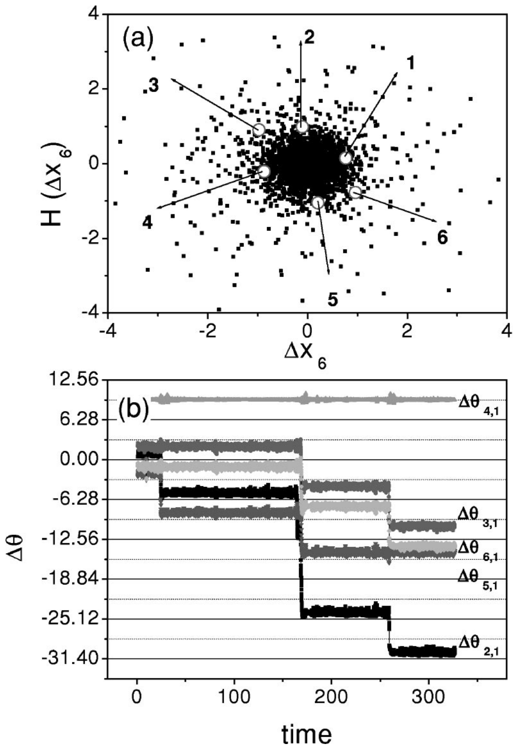

FIG. 3. (a), (b) The same as Figs. 2(a) and (b), respectively, with $e=9.0, r=3.7$ [square in Fig. 1(a)]. A stick-split behavior in the motions of $\Delta\theta_{j,1}(t)$ is shown in (b).

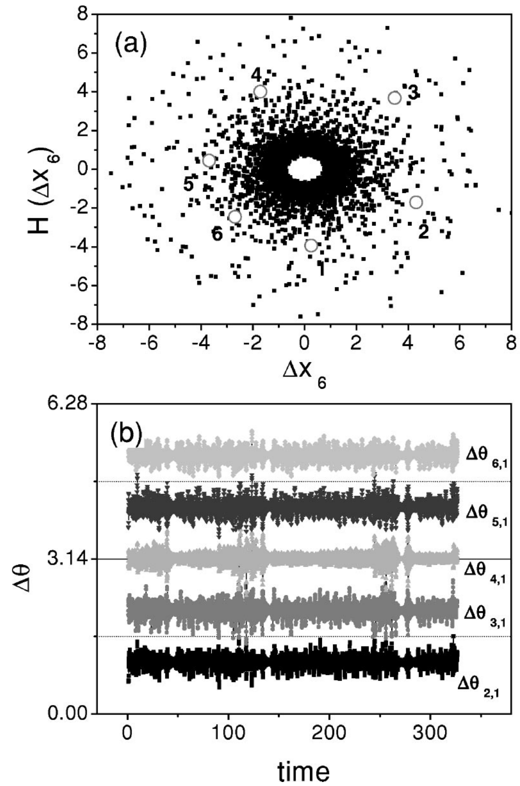

FIG. 4. (a), (b) The same as Figs. 2(a) and (b), respectively, with $e=15.1, r=6.0$ [triangle in Fig. 1(a)]. Phase locking between $\theta_j(t)$ is clearly identified.

tain values $\Delta\theta_{j,1}(t)\approx n\pi, n=0,\pm 1,\pm 2,\ldots$ for a long time, and then randomly jump a π or 2π angle to other values. There is no phase locking between various $\theta_j(t)$.

In Figs. 3(a) and 3(b), we use parameters $e=9.0, r=3.7$ [see the hollow square in Fig. 1(a)], and do the same as Figs. 2(a) and 2(b), respectively. The scattering points show on-off intermittency in Fig. 3(a) similar to Fig. 2(a). However, the snapshot of the distribution of the six oscillators on the $\Delta x_j - H(\Delta x_j)$ plane at an arbitrary time show clearly an ordered phase distribution:

$$\Delta\theta_{j,1}(t)\approx 2n\pi+\frac{2\pi}{6}(j-1)$$

$$n=0,\pm 1,\ldots, \quad j=2,3,\ldots,6. \tag{6}$$

This phase order is not always kept, and abrupt bursts can break this order from time to time. In the evolutions of $\Delta\theta_{j,1}(t)$ of Fig. 3(b), we observe basically the phase relation (6) and some abrupt breakings of this ordering show typical stick-split motions, which are well known to be a clear precursor of phase locking [10].

In Figs. 4(a) and 4(b), we take the parameter set $e=15.1, r=6.0$ [denoted by the triangle in Fig. 1(a)], and, again, plot the figures same as Figs. 2(a) and 2(b), respectively. With considerably large gradient coupling r, Fig. 4(a) manifests some features essentially different from Fig. 2(a), though the random scattering points showing intermittent bursts remain unchanged. The snapshot of the distribution of the six oscillators in Fig. 4(a) demonstrates the phase ordering of Eq. (6). This is similar to Fig. 3(a). However, unlike Fig. 3(b), there are no burst breaks of this phase ordering in Fig. 4(b). In Figs. 4(a) and 4(b) phase locking is apparently observed.

From Fig. 2 to Fig. 4 there must be a bifurcation from a phase unlocking state to a phase locking state. In order to identify this bifurcation point we plot in Figs. 5(a), 5(b), and 5(c) $\ln[V(t)]$ vs $\ln(t)$ for the three cases of Figs. 2, 3, and 4, respectively, with the average phase variance $V(t)$ defined as

$$V(t)=\frac{1}{5}\sum_{j=2}^{6}\left[\langle\Delta^2\theta_{j,1}(t)\rangle-\langle\Delta\theta_{j,1}(t)\rangle^2\right]. \tag{7}$$

The plots obey the following nice power law:

$$V(t)\propto t^D. \tag{8}$$

In Figs. 5(a) and 5(b), $V(t)$'s increase linearly on t, $D\approx 1$, indicating random walks of the phase motion. In Fig. 5(c), $V(t)$ fluctuates in a constant value and the zero slope of the plots, $D=0$, confirms phase locking status. In Fig. 5(d) we examine different points (denoted by different r) on the bifurcation curve of Fig. 1(a) (for saving computing time we choose points slightly above the curve), and measure the slopes of the corresponding $\ln[V(t)]-\ln(t)$ curves, D's, against r. A sharp dropdown of D from nearly $D=1$ to zero at the point $r_c\approx 5.0$ in our case indicates that a bifurcation from phase unlocking to phase locking occurs.

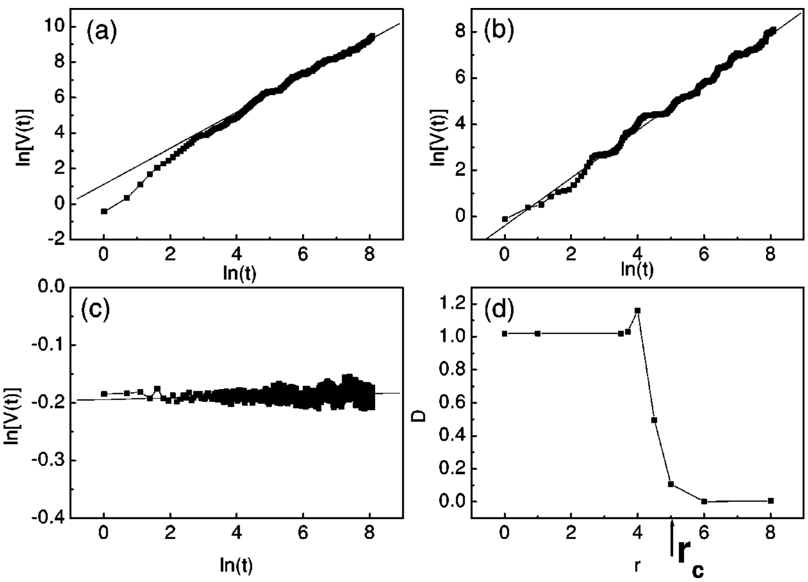

FIG. 5. (a), (b), (c). ln[$V(t)$] plotted vs ln(t) for the motions at circle, square, and triangle of Fig. 1(a), respectively, where the variance $V(t)$ is given in Eq. (7). In (a) and (b), $V(t)$ increases linearly with t, $D\approx 1$, confirming random walk behavior of $\Delta\theta_{j,1}(t)$, while the practically constant $V(t)$ in (c) indicates a phase-locking status. (d) The slope of the ln[$V(t)$]$-$ln(t) curve, D, plotted vs r. All the slopes are computed along the bifurcation line of Fig. 1(a) (slightly above the line). The phase locking bifurcation on the desynchronization bifurcation line occurs at $r=r_c\approx 5.0$.

In conclusion, we have investigated the characteristic changes of on-off intermittency in the synchronization-desynchronization bifurcation line in coupled chaotic oscillators with both symmetric and asymmetric couplings. It is shown that by varying certain control parameters the on-off intermittencies of different oscillators can change from phase-unlocking status to phase-locking status through a stick-split phase relation. This phase-locking transition occurs in a bifurcation line of instability of the synchronous chaos. Thus, this transition can be regarded as a codimension-two bifurcation of coupled chaotic oscillators. An important point is that the new phase-locking transition cannot be directly observed in the original variable space, i.e., the space of $[x_j(t),y_j(t),z_j(t)]$, and it can be identified only in the variable difference space, i.e., the space of $[\Delta x_j(t),\Delta y_j(t),\Delta z_j(t)]$. The reason is that in the $[x_j(t),y_j(t),z_j(t)]$ space, the synchronous chaotic motion is not subtracted, and with this element the equal-separation phase distribution of Fig. 4 can never be observed. Moreover, the phenomena found in this paper may be generally observed in coupled chaotic systems. Very rich behaviors, including multicodimension bifurcations, are expected to be possible at the instability of synchronous chaos, if increasing number of control parameters are available. This is an interesting field worthwhile investigating.

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